Perrault's watch and Beltrami's pseudosphere.
A story without a moral

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1.

If a specter haunted 19th century mathematics, it was the specter of the pseudosphere, i.e. the two-dimensional space with constant negative curvature.

Already toward the end of the previous century, Johann Heinrich Lambert (1728-1777), in his original investigation about Euclid's fifth postulate,[1] hinted that the angles of a triangle could sum to less than two right angles in the case where the triangle lies on an "imaginary sphere" (*imaginäre Kugelfläche*) [Lambert 1895, p. 203]. Lambert's highly speculative remark was suggested by the «analogie entre le cercle et l'hyperbole» he had already exploited in his work on the irrationality of pi [Lambert 1768]; however, it would have been impossible to express that insight in a mathematically more definite form simply because, at the time, essential geometrical notions – first of all, that of curvature – were still to be developed.

Those notions were introduced (or should I say "invented"?) in Gauss's *Disquisitiones generales circa superficies curvas* [Gauss 1827], a groundbreaking memoir whose methods and ideas were deeply influenced by its author's extensive work in geodesy and cartography.[2] Though only surfaces embedded in three-dimensional Euclidean space were considered, Gauss succeeded in pinpointing two intrinsic geometric quantities: the

[1] *Theorie der Parallellinien*, 1776, published posthumously 1786; [Lambert 1895].
[2] Cf. [Breitenberger 1984] and, more generally, [Bühler 1981].

first fundamental form[3] and the *mensura curvaturae*, that we now call "Gaussian curvature". Now, in Gauss's quite exhaustive treatment of the subject there is a noticeable blank, an apparent gap: whilst the Euclidean plane and the round sphere of radius R have, respectively, Gaussian curvature 0 and $1/R^2$, there is no instance of a surface having constant negative Gaussian curvature. It is hard to believe that such a conspicuous asymmetry could have escaped Gauss's attention. Actually, in an unpublished note dating from 1823 to 1827, Gauss studied the surface created by rotating the plane curve known as "tractrix" around its own asymptote; the resulting surface of revolution was christened "the opposite of the sphere" (*das Gegenstück der Kugel*) [Gauss 1900, pp. 264-265]. It is not at all a difficult exercise in differential geometry to check that this surface has constant negative Gaussian curvature; however, this result does not appear in Gauss's note, in spite of the fact that he could have derived it as a straightforward consequence of the theorems proved in the *Disquisitiones generales circa superficies curvas*. Then, how can the name "das Gegenstück der Kugel" be accounted for? Could it be conjectured that Gauss's surprising laziness in computing the *mensura curvaturae* of this surface somehow reflects his reluctance to disclose his more heterodox ideas about non-Euclidean geometry? Needless to say, cautious historians are wise enough to suspend judgment on questions of this kind.[4]

2.

Tractrix can be defined as the curve in the x, y plane such that the tangent segment between the contact point and the x axis has constant length (fig. 1). Hence, the problem of determining such a curve from some initial data represents, in the language of 17th century mathematics, an example of "inverse problem of tangents". This amounts to say, in today's parlance, that the solution of the problem can be obtained by integrating a first-order differential equation. In the case of tractrix, if one assumes that y takes some

[3] Very roughly speaking, the first fundamental form is an inner product defined on tangent vectors, which allows one to measure curve lengths and angles between directions.

[4] Cf. [Gray 2007, p. 124].

positive value a for $x = 0$, the corresponding differential equation can be written in the form:

$$dx/dy = \pm (a^2 - y^2)^{½}/y.$$

The integration of this equation between 0 and x yields the equation of the tractrix in Cartesian coordinates:

$$x = \pm \{(a^2 - y^2)^{½} - a \log [(a + (a^2 - y^2)^{½})/y]\}$$

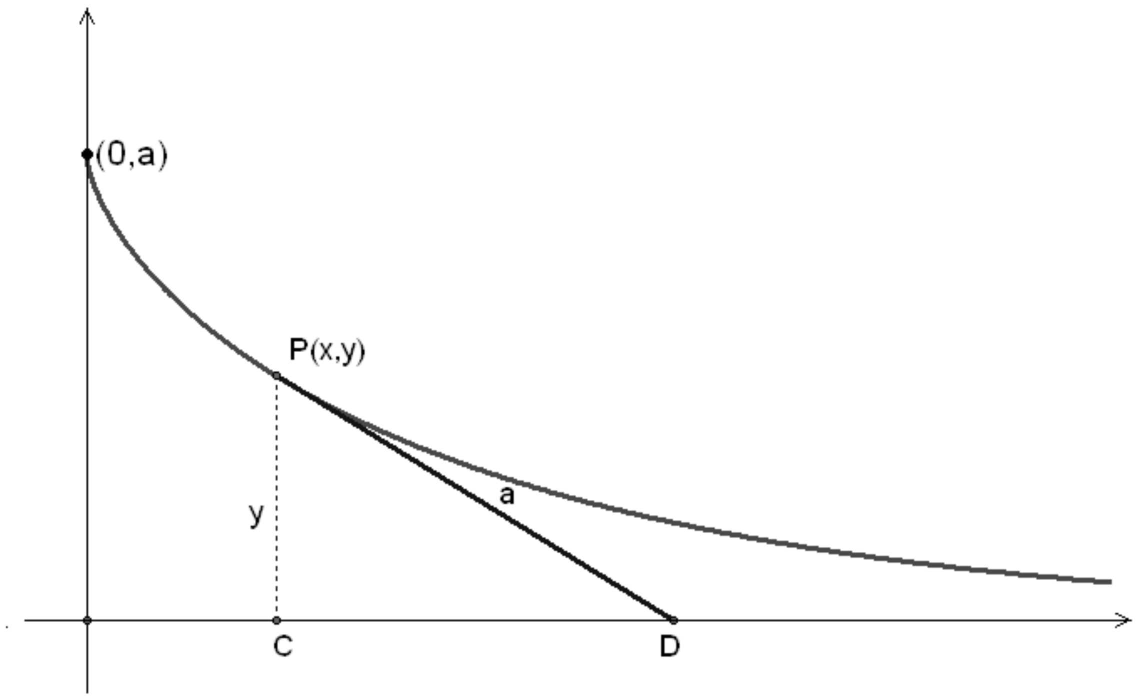

Figure 1 The positive branch of the tractrix

The inverse problem of tangents is a crucial issue in Newton's second letter to Leibniz (through care of Henry Oldenburg), the so-called *Epistola Posterior*, dated October 24, 1676 (O.S.).[5] In polemical reply to Leibniz's remarks about the *Epistola Prior*, Newton makes the following comment:

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[5] For a detailed analysis of the *Epistola Posterior* we refer to [Scriba 1963] and [Guicciardini 2009, pp. 356-360].

> [...] inverse problems of tangents are within our power, and others more difficult than those, and to solve them I have used a twofold method of which one part is neater, the other more general. At present I have thought fit to register them both by transposed letters, lest, through others obtaining the same result, I should be compelled to change the plan in some respects. 5accdæ10effh11i4l3m9n6oqqr8s11t9v3x: 11ab3cdd10eæg10ill4m7n6o3p3q6r5s11t8vx, 3acæ4egh5i4l4m5n8oq4r3s6t4vaaddæeeeeeiijmmnnooprrrsssssttuu [Newton 1960, p. 148].

Just after this anagrammatic statement Newton added a few worlds about the tractrix, even though he did not make use of any special term to designate it:

> This inverse problem of tangents, when the tangent between the point of contact and the axis of the figure is of given length, does not demand these methods. Yet it is that mechanical curve the determination of which depends on the area of an hyperbola [*ibid.*, p.148].

The expression "mechanical curve" (*curva mechanica*) is to be referred to the distinction introduced by Descartes in his essay *La Géométrie*, which was published as an appendix to *Discours de la méthode* (1637). Descartes differentiates between admissible curves "qu'on peut recevoir en géométrie" and curves that are to be banished from geometry. The latter – according to Descartes, who criticizes the definition given by the "anciens géomètres" – are the "mechanical curves" that are "décrites par deux mouvements séparés, et qui n'ont entre eux aucun rapport qu'on puisse mesurer exactement" [Descartes 1982, p. 390]; instances of such curves are the Archimedean spiral and the quadratrix. By taking advantage of the new coordinate method, admissible curves can be characterized as those given by an algebraic equation, while mechanical curves are found to coincide with transcendental curves [*ibid.*, p. 392].

Ironically, Descartes's proscription of mechanical curves from the realm of geometry appeared strikingly artificial just within the theoretical framework he contributed to shape. Moreover, it was in strong contrast with the early development of infinitesimal calculus, where the study of non algebraic curves (e.g. the logarithmic curve) was one of the most challenging research topics. In the long-running process of "legitimation of transcendental curves" [Bos 1988, 1997] a prominent role was played by curves generated by "tractional motion", i.e. traced by dragging a heavy body over a horizontal plane: indeed, depending on several factors such as the shape of the dragged object, the friction encountered during motion, etc., the tractional curve can turn to be algebraic or transcendental.

The first appearance of the tractrix on the European scientific scene was not in Newton's *Epistola Posterior* but, a few years earlier, in Parisian salons: it originated as a tractional curve. In the first half of 1670s, Claude Perrault (1613-1688) – Charles's brother, a talented and versatile *savant*, active in such different fields as zoology, medicine, plant and animal physiology, architecture, classical philology, mechanical engineering – amused himself challenging mathematicians with a problem which required, so to say, a little theatrical staging. Perrault would take out his watch, that was contained in a silver case, put it on a table and pull the end of the watch-chain along a straight line; then, he would point-blank ask his puzzled interlocutor: "Monsieur, what is the geometrical curve traced by my watch on the plane of the table?"

It is not difficult to recognize that the curve described by Perrault's watch must be a tractrix: actually, the watch-chain has constant length and its direction is tangent to the trajectory due to the relatively high friction between the case and the table surface. The solution to the problem – not taking into consideration Newton's priority claim[6] – was found out, independently, by Leibniz and Christiaan Huygens (1629-1695).[7] It is Leibniz who – in a memoir published in *Acta Eruditorum* in September 1693[8] – tells us the story

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6 When Newton asserts that the problem "does not demand [the] methods" of calculus, he can only mean that he is able to solve it by means of series expansions. I am unaware of any evidence among Newton's paper that shows that he actually performed this computation.

7 In order to avoid possible confusion with the world "solution", it should be stressed that neither Leibniz, nor Huygens wrote down the Cartesian equation or a parametric representation for the tractrix. Both of them showed that the problem can be reduced to the "quadrature of the hyperbola", or equivalently – in modern terms – to the computation of an integral of the type $\int dx/x = \log x + C$ (actually, the integral to be solved is somehow more complicated than that, but its primitive function involves the logarithm). Moreover, both of them designed mechanical devises to draw tractrix curves.

8 "Supplementum geometriae dimensioriae seu generalissima omnium tetragonismorum effectio per motum: similiterque multiplex conscructio lineae ex data tangentium conditione" = [Leibniz 1858, pp. 294-301]; see also [Leibniz 1989, pp. 247-267].

of Perrault's problem[9] (including the detail of the "horologium portatile suae thecae argenteae inclusum") and claims that he solved it more than eighteen years earlier.[10] Huygens had been staying in Paris for an almost uninterrupted period from 1666 through 1681, so that it is very likely he was aware of Perrault's conundrum. In a memoir published, in 1692-1693, in the form of a letter to the historian Henri Basnage de Beauval, the great Dutch scientist determined several fundamental properties of the curve he called *tractoria*.[11] In particular, he studied the solid of rotation that is created by revolving this curve around its asymptote. If a is the (constant) length of the tangent segment between the contact point and the x axis, Huygens was able to prove that the volume of this "infinite solid" is equal to a quarter of that of the sphere of radius a (namely, $\pi a^3/3$) and that its lateral area is equal to that of a circle of radius $a\sqrt{2}$ (namely, $2\pi a^2$) [Huygens 1905, p. 409].

3.

The same surface of revolution studied by Huygens and, some hundred and fifty years later, by Gauss, happens to be encountered in an article by Ferdinand Minding (1806-1885) published in the *Journal für die reine und angewandte Mathematik* in 1839. Minding – who studied philosophy, physics and philology at the universities of Halle and Berlin, but was essentially self-taught in mathematics – is mainly remembered for his

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9 Leibniz met Claude Perrault during his *séjour* in Paris, from 1672 through 1676, but their acquaintance must be rather limited; cf. [Hofmann 1974]. However, in the first months of 1676, Leibniz sent Perrault a detailed criticism of a memoir about gravitation the latter had presented to the Académie in 1669; cf. [Aiton 1991, p. 75].

10 Leibniz specifies that his solution has remained "ultra Horatiani limitis duplum pressa", referring to the well-known passage in the *Ars poetica* (vs. 388) which urges that poems should be kept back for nine years: "si quid tamen olim scripseris, nonum prematur in annum".

11 Huygens's memoir appeared in the *Histoire des ouvrages des sçavans* issues of December 1692 and February 1693; it is reproduced in [Huygens 1905, pp. 407-417]. In a letter of January 12, 1693, Huygens wrote to Leibniz about his results on the tractrix: "Jes vous entretiendray une autre fois d'une quadrature physico-mathematique de l'Hyperbole que j'ay rencontré il n'y a guere, dont la speculation a quelque chose de plaisant" [Huygens 1905, p. 388].

contributions to differential geometry.[12] Gauss's *theorema egregium* (proved in the *Disquisitiones generales circa superficies curvas*) implies that, if two surfaces are "developable upon each other",[13] then their Gaussian curvatures are equal. The converse of this result is, in general, false: Gaussian curvature does not uniquely determine the first fundamental form.[14] However, as Minding proved in his 1839 article, if two surfaces has constant Gaussian curvature and their curvatures are equal, then they are developable upon each other. In order to illustrate his result, Minding supplied some novel examples of surfaces of negative constant Gaussian curvature: among them there is – elegantly parametrized – the surface of revolution generated by the tractrix (fig. 2), to which no special name was assigned.

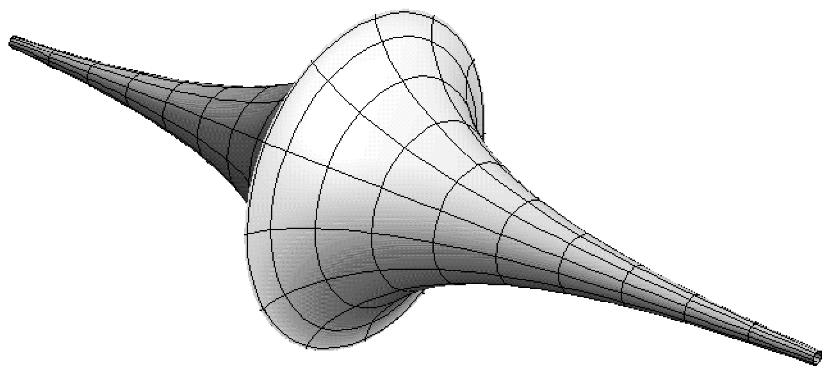

Figure 2

[12] Cf. [Youschkevitch 1990] and [Reich 1973].

[13] In modern differential geometry one would say that the two surfaces are "locally isometric". This means that it is possible to map any sufficiently small region of one surface to a corresponding region of the second preserving both lengths and angles.

[14] It is intuitively obvious that Gaussian curvature of a surface, being just one function, cannot be enough to determine its first fundamental form, which is locally given by the assignment of three functions. All geometric information contained in a surface embedded in Euclidean space can be proved to be encoded in its first and second fundamental forms; the coefficients of the second fundamental form are related to the coefficients of the first fundamental form through relations already derived by Gauss (and equivalent to the *theorema egregium*) and satisfy the so-called Mainardi-Codazzi equations. The far more difficult problem of characterizing isometries between two abstract Riemannian surfaces will be solved by Élie Cartan in the second edition of his masterly *Leçons sur les espaces de Riemann* (second ed. 1946); see [Berger 2003, p. 214].

The following year, in a short paper once again published in the *Journal für die reine und angewandte Mathematik*, Minding found out a quite remarkable relation: by interchanging hyperbolic and trigonometric functions, the formulas for the resolution of geodesic triangles on a surface of negative constant curvature $1/R^2$ are transformed into the formulas for the resolution of geodesic triangles on a sphere of radius *R*. In this way, Lambert's vague analogy "entre le cercle et l'hyperbole" between had eventually found a precise formulation in geometrical terms, as well as Gauss's "opposite of the sphere" its *raison d'être*. But that was not all. Only three years earlier, in a visionary paper published in the same journal, the Russian mathematician Nicolaj Ivanovič Lobačevskij had reached a conclusion analogous to Minding's with regard to trigonometric formulas for triangles of his "géométrie imaginaire" [1837, pp. 299-300]. Apparently unaware of this result, Minding missed the chance to be the first to devise a model of non-Euclidean (hyperbolic) geometry with the help of differential geometric tools.

The results in Minding's 1840 paper were re-derived in 1857 by an obscure teacher at the Liceo of Pavia, Delfino Codazzi (1824-1873), who took as his starting point Liouville's forth addition [1850] to Gaspard Monge's *Application de l'analyse à la géométrie*, where a complete classification of surface of constant curvature was given.[15] Codazzi "espon[e] la risoluzione di un triangolo qualunque formato [...] mediante l'intersezione di tre geodetiche [sulle superficie] che possono adattarsi a quella di rivoluzione generata dalla linea avente le tangenti di lunghezza costante" [Codazzi 1857, p. 346]; of course, this "linea" is the tractrix. In the author's own words,, the analogy of the resulting formulas with those of spherical trigonometry "balza agli occhi". Yet again, and unsurprisingly, no connection with non-Euclidean geometry is established.

[15] Liouville's Note IV is a true masterpiece of mathematical clarity, whose influence on the following developments of differential geometry has been largely underestimated.

4.

The mathematician who happened to have the insight and the ability to put together the pieces of the puzzle was Eugenio Beltrami (1835-1900).[16] After having been appointed professor of geodesy at the University of Pisa in 1864, he started doing research work in cartography and published a pioneering article whose long title is self-explaining: "Risoluzione del problema: 'riportare i punti di una superficie sopra un piano in modo che le linee geodetiche vengano rappresentate da linee rette'" [Beltrami 1865]. In 1866 Beltrami assimilated the main notions of non-Euclidean geometry through Jules Hoüel's translation of Lobačevskij's *Geometrische Untersuchungen*, which appeared with the addition of several letters from Gauss and Schumacher's correspondence.[17] Combining these new ideas with differential geometric techniques and cartographic methods, he was able, in a rather short time, to complete a deeply innovative paper, "Saggio di interpretazione della geometria non euclidea", that appeared in Giuseppe Battaglini's *Giornale di matematiche* in 1868.[18] In the introduction the author's methodological standpoint is clearly expounded:

> In questi ultimi tempi il pubblico matematico ha incominciato ad occuparsi di alcuni concetti i quali sembrano destinati, in caso che prevalgano, a mutare profondamente l'ordito della classica geometria. [...] Siffatti tentativi di rinnovamento radicale dei principî si incontrano non di rado nella storia dello scibile. [...] D'altronde nella scienza matematica il trionfo di concetti nuovi non può mai inifrmare le verità già acquisite: esso può soltanto mutarne il posto o la ragion logica, e crescerne o scemarne il pregio e l'uso. Né la critica dei principî può mai nuocere alla solidità dell'edificio scientifico, quando pure non conduca a scoprirne e riconoscerne meglio le basi vere e proprie. Mossi da questi intendimenti noi abbiamo cercato, per quanto le nostre forze lo consentivano, di dar ragione a noi stessi dei risultati a cui conduce la dottrina di Lobačevskij; e, seguendo un processo che ci sembra in tutto conforme alle buone tradizioni della ricerca scientifica, abbiamo tentato di trovare un substrato reale a

[16] See [Loria 1937, pp. 169-209] (originally published in *Bibliotheca mathematica* (2), 3 (1901), pp. 392-440) and [Tazzioli 2000].

[17] Nikolaj Ivanovič Lobačevskij, "Études géométriques sur la théorie des parallèles", traduit de l'allemand par Jules Hoüel; suivi d'un extrait de la Correspondance de Gauss et Schumacher, *Mémoires de la Société des sciences physiques et naturelles de Bordeaux*, 4 (1866), pp. 83-128.

[18] The delay in publication was due to an objection made by his mentor Luigi Cremona (actually, the objection was groundless and naïve).

> quella dottrina, prima di ammettere per essa la necessità di un nuovo ordine di enti e di concetti [Beltrami 1868, pp. 374-375].

Beltrami's conclusion is unequivocal.[19] There is no need of new entities or new concepts, since "i teoremi della planimetria non-eculidea sussistono incondizionatamente per tutte le superficie di curvatura costante negativa" [*ibid.*, p. 377]. Therefore, surfaces of negative constant curvature – that are collectively named by Beltrami "pseudosferiche"[20] – provide the "substrato reale" for hyperbolic geometry.

Soon after the publication of Beltrami's "Saggio", a number of mathematicians expressed doubts about the actual solidity of the asserted "substrato reale". It is really possible – they wondered – to construct a regular[21] surface embedded in the Euclidean space that provides a faithful and global representation of Lobačevskij-Bolyai plane? Quite obviously, Beltrami's pseudosphere does not satisfy this requirement: not only for quite inessential topological reasons,[22] but also because it ends abruptly with a ridge, so that it is not complete.[23] The great German scientist Hermann von Helmholtz, in an expository paper published in 1870,[24] gave a vivid description of pseudospherical surfaces:

> So, for example, a pseudospherical surface may appear in the shape of a ring, the inner surface of which is convex to the axis. Or it may appear like the outer surface of champagne-glass, the stem widening out upwards into an outward curved margin, and prolonged downwards into an infintely long and thin thread [Helmholtz 1870, p. 129].

With his usual perspicacity, Helmholtz explained the core of the problem:

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19 For an analysis of Beltrami's paper see [Stillwell 1996] and [Bartocci ??].

20 In Beltrami's "Saggio" three different kinds of pseudospherical surfaces are studied; among them, the surface of rotation generated by a branch of the tractrix is usually called Beltrami's pseudosphere.

21 Here "regular" means "without singular points".

22 The pseudosphere has the topology of a cylinder, not of a plane. This issue is inessential because it would be enough to make a cut in the surface (as suggested by Beltrami himself) in order to obtain a space with the topology of the plane.

23 A topological space is said to be complete if any Cauchy sequence is convergent; this is a technical and subtle point.

24 This article, "The axioms of geometry", appeared in the issue of February 12, 1870 of *The academy. A Monthly Record of Literature, Learning, Science, and Art* and was an English reduction of the *Docentenverein* address "Über den Ursprung und die Bedeutung der Geometrische Axiome" that Helmholtz had delivered in Heidelberg in January 1870.

> we cannot in our space construct any pseudospherical surface, infinitely extended in the direction of the axis of revolution. We always arrive either at one limit, as in the case of a champagne-glass, or at two limits, as in the case of a ring. At those limits the smaller radius of curvature becomes evanescent, the greater infinite [*ibid.*, p. 129].

Helmholtz's perplexities were shared by several other mathematicians of the time, among whom Felix Klein[25] and Angelo Genocchi.[26]

At the beginning of the new century, in 1900, David Hilbert proved that a complete, regular surface of constant negative curvature cannot be embedded in the Euclidean space.[27]

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[25] According to Klein, Beltrami's "Interpretation bringt leider [...] nie das gesamte Gebiet der Ebene zur Anschauung, indem die Flächen mit konstantem negativen Krümmungsmasse wohl immer durch Rükkehrkurven usw. begrenzt werden" [Klein 1870 =1923, p. 247].

[26] Genocchi wrote: "è dunque possibile, per non dire probabile, che l'integrale dell'equazione alle derivate parziali che esprime la superficie a curvatura costante non presenti che una sola superficie reale, infinita in tutte le direzioni e semplicemente connessa, vale a dire, il piano [...] euclideo, la cui curvatura è ovunque nulla" [Genocchi 1873, p.188].

[27] See [Hilbert 1901].

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